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Quantum Circuits and Schrödinger's Cat States

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Abstract Quantum systems that are confined to circuit geometries are called quantum circuits. Macroscopic superconducting circuits are quantum circuits which can be modelled using a *Quantisation by Parts* scheme based on the macroscopic wave function approach of Feynman. This paper studies the circuit composed of an input wire and an output plate. We find that in order to achieve a consistent theory of supercurrent flow we have to generalize the quantisation by parts scheme to quantise in a path space. The generalized theory predicts a current flow down the wire into the plane. In addition to a current flowing radially outwards in the plane, the theory allows a circulating current round the origin. Strikingly, the circulating current can flow clockwise or anti-clockwise in such a way as to generate a magnetic moment of magnitude half of a Bohr magneton for an orbiting electron in an atom and a magnetic flux half that of the magnetic flux quantum of a superconducting ring. There is also the possibility of a macroscopic superposition of the two states of opposing circulating currents resembling a Schrödinger's cat situation. Furthermore, we outline a setup involving an external magnetic field that may allow experimental tests of the theory.

Keywords Quantum circuits · Quantisation

1 Introduction

We present an application of the *Quantisation by Parts* scheme based on a macroscopic wave function approach to a two-branch quantum circuit consisting of a superconducting wire connecting to a superconducting plane. The macroscopic wave function approach to superconductivity was championed by Feynman [1], and a quantisation by parts scheme was introduced by Wan *et. al.* to establish a systematic theory for such an approach [2–7]. The scheme is based on the mathematical treatment of point interactions studied by a number of people [8–10]. A circuit, be it classical or quantum, composes of continuous segments called *branches*. These branches are connected to form a circuit. A point where

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branches are connected is known as a *branch point* or a *junction*. For a quantum circuit the branches may not have direct physical contact at a branch point; a quantum current can tunnel through the branch point. The geometry may alter abruptly at a branch point, e.g., a single branch may be connected to two other branches down stream. This prevents any traditional quantisation method which treats the geometry of the whole system as a Riemannian manifold to proceed. The method of quantisation by parts has proved to be applicable to a variety of circuit configurations. The idea is to treat the geometry of each branch as a Euclidean space on which we can carry out an initial quantisation. A quantum theory for the whole circuit can then be obtained by suitably combining all the individually quantised branches together. More specifically the scheme consists of three quantisation stages [7]:

1. *Partial Quantisation* Each branch of the circuit is associated an appropriate Hilbert space of square integral functions over the branch, be it a line or a plane. The operators representing the observables of interest on each branch during partial quantisation are not required to be selfadjoint; it is sufficient to be symmetric.
2. *Composite Quantisation* This involves the construction of the Hilbert space and observables for the entire circuit. The Hilbert space of the whole circuit is taken to be the direct sum of the individual Hilbert spaces:

$$\mathcal{H}^c = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots. \tag{1}$$

   We shall denote composite quantities for the system as a whole by the superscript c while those for partial quantisation for each branch are denoted by a numerical subscript. Observables for the circuit system as a whole are taken to be appropriate selfadjoint or maximal symmetric extensions of the direct sums of the corresponding partially quantised operators.[1] This process usually results in non-uniqueness as there may be an infinite number of extensions for each observable. This turns out to be of crucial importance as this enables us to deal with a variety of circuits. Each particular circuit configuration would corresponds to a particular set of extensions, chosen by considering the physics involved. Selfadjoint extensions of symmetric operators are often specifiable by boundary conditions which make things a lot more transparent; a summary of the technicalities involved may be found in [7] (pp. 113–130, pp. 209–212), [8, 9].
3. *Correlative Quantisation* For a given system we need to correlate certain observables in order to produce the desired physical properties. This is achieved by correlating the selfadjoint extensions of the corresponding operators based on some physical assumptions about the nature of the system. For instance we require the compatibility of the Hamiltonian and the momentum of a superconducting circuit, in the sense that the two observables must share common eigenfunctions, in order to maintain a persistent dc supercurrent observed in experiments [7] (p. 519).

For the specific case of a superconducting wire linked to a superconducting plane, as shown diagrammatically in Fig. 2 in [8, 9], we want to see what would happen if a supercurrent is fed into the wire, e.g., how the current will flow into the superconducting plane. We are particularly interested to see if anything new, both in terms of the fundamental theory of quantisation by parts and in terms of any new physical effects, emerges from such a circuit configuration. In Sect. 2 we detail the traditional approach quantising on the physical space,

[1] The reasoning for incorporating maximal symmetric operators in the representation of quantum observables is available in [7] (p. 403).

and explain the need to generalize to quantisation on *path spaces*. Section 3 provides a brief introduction to the concepts of path spaces and of Hilbert spaces on path spaces, followed by quantisation in a two-dimensional plane with a hole. In Sect. 4 we extend the quantisation to the circuit of a superconducting plane linked to a superconducting wire. Section 5 demonstrates the possible existence of a Schrödinger's cat state on this circuit. In Sect. 6 we outline a possible experimental setup to test for this state.

2 Quantisation in the Physical Space $\mathbb{E}^- \times \mathbb{E}_h^2$

2.1 Hilbert Spaces on Physical Spaces

A two-branch circuit formed by a superconducting wire connected to a superconducting plane can be realized mathematically by idealizing the wire (first branch) as a half-line, i.e., the one-dimensional Euclidean space $\mathbb{E}^- = \{x \in (-\infty, 0)\}$ with volume element dx,[2] and the plane (second branch) as a Euclidean plane $\mathbb{E}_h^2$ with a hole at the origin, i.e., with its origin removed. Physically one would imagine the current to flow through the surface of the wire; the current would then flow directly into a small circle of diameter of the wire round the origin, not through the origin of the plane, and then spread out into the plane. As will be seen later, the removal of the origin will make a fundamental difference to the path space formulation of our theory.

We shall denote by $C_0^\infty(\mathbb{E}^-)$ the set of infinitely differentiable functions of compact support on $\mathbb{E}^-$. Functions on $\mathbb{E}_h^2$ may be dependent on the usual polar coordinates r and θ. Functions depending only on r are regarded as functions on $\mathbb{R}^+ = \{r \in (0, \infty)\}$, while functions depending only on θ are regarded as functions on $\mathcal{C}$, the circle of unit radius centered at the origin of the plane. The notation $\mathbb{R}^+$ signifies that the set $\{r \in (0, \infty)\}$ does not form a Euclidean space; the volume element here is rdr. The symbol $C_0^\infty(\mathbb{R}^+)$ denotes the set of infinitely differentiable functions of compact support on $\mathbb{R}^+$. Furthermore $L^2(\mathbb{E}^-)$ and $L^2(\mathbb{E}_h^2)$ shall denote the Hilbert spaces formed by square-integrable functions on $\mathbb{E}^-$ and $\mathbb{E}_h^2$ respectively. Also we shall adopt the following notations:

$$\mathcal{H}(\mathbb{E}^-) = L^2(\mathbb{E}^-) = \left\{\phi_1(x) : \int_{-\infty}^0 \phi_1^*(x)\phi_1(x)dx < \infty\right\}, \tag{2}$$

$$\mathcal{H}(\mathbb{R}^+) = L^2(\mathbb{R}^+, rdr) = \left\{\phi_2(r) : \int_0^\infty \phi_2^*(r)\phi_2(r)rdr < \infty\right\}, \tag{3}$$

$$\mathcal{H}(\mathcal{C}) = L^2(\mathcal{C}, d\theta) = \left\{\eta(\theta) : \int_0^{2\pi} \eta^*(\theta)\eta(\theta)d\theta < \infty\right\}, \tag{4}$$

$$\mathcal{H}(\mathbb{E}_h^2) = L^2(\mathbb{E}_h^2) = L^2(\mathbb{R}^+, rdr) \otimes L^2(\mathcal{C}, d\theta), \tag{5}$$

$$\mathcal{H}^c = L^2(\mathbb{E}^-) \oplus L^2(\mathbb{E}_h^2). \tag{6}$$

The subscripts 1, 2 signify functions on branches 1 and 2 respectively. Equation (5) indicates that $\mathcal{H}(\mathbb{E}_h^2)$ can be decomposed into a direct product of a radial part and an angular part. The identity operators on $\mathcal{H}(\mathbb{E}^-)$, $\mathcal{H}(\mathbb{R}^+)$ and $\mathcal{H}(\mathcal{C})$ are denoted by $\widehat{\mathbb{I}}(\mathbb{E}^-)$, $\widehat{\mathbb{I}}(\mathbb{R}^+)$ and $\widehat{\mathbb{I}}(\mathcal{C})$ respectively. Furthermore, the zero elements in $\mathcal{H}(\mathbb{E}^-)$, $\mathcal{H}(\mathbb{E}_h^2)$, $\mathcal{H}(\mathbb{R}^+)$ and $\mathcal{H}(\mathcal{C})$ are

[2] The real line as a Euclidean space is denoted by $\mathbb{E} = \{x \in (-\infty, \infty)\}$.

denoted by $0(\mathbb{E}^-)$, $0(\mathbb{E}_h^2)$, $0(\mathbb{R}^+)$ and $0(\mathcal{C})$ respectively so that, for example, an element $\phi_1(x) \in \mathcal{H}(\mathbb{E}^-)$ can be extended to $\mathcal{H}^c$ as $\phi_1(x) \oplus 0(\mathbb{E}_h^2)$. For clarity we shall denote the zero operators in these spaces by $\widehat{0}(\mathbb{E}^-)$, $\widehat{0}({\mathbb{E}_h}^2)$, $\widehat{0}(\mathbb{R}^+)$ and $\widehat{0}(\mathcal{C})$ respectively.

Let $\widehat{L}(\mathcal{C})$ denotes the selfadjoint operator in $\mathcal{H}(\mathcal{C})$ defined by the operator expression $-i\hbar\partial/\partial\theta$ acting on the domain of absolutely continuous functions $\eta(\theta)$ on $\mathcal{C}$ satisfying the usual periodic boundary condition [7] (pp. 75–77, p. 480):

$$\eta(\theta) = \eta(\theta + 2\pi). \tag{7}$$

$\widehat{L}(\mathcal{C})$ admits the following eigenfunctions

$$\eta_n(\theta) = e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \dots, \tag{8}$$

corresponding to eigenvalues $n\hbar$. We shall denote the eigensubspaces corresponding to eigenfunctions $\eta_n(\theta)$ by $\mathcal{L}_n(\mathcal{C})$. We can decompose $\mathcal{H}(\mathcal{C})$ into a direct sum of $\mathcal{L}_n(\mathcal{C})$:

$$\mathcal{H}(\mathcal{C}) = \oplus_n \mathcal{L}_n(\mathcal{C}). \tag{9}$$

Let

$$\mathcal{H}_n(\mathbb{E}_h^2) = \mathcal{H}(\mathbb{R}^+) \otimes \mathcal{L}_n(\mathcal{C}). \tag{10}$$

We then have

$$\mathcal{H}(\mathbb{E}_h^2) = L^2(\mathbb{E}_h^2) = \oplus_n \mathcal{H}_n(\mathbb{E}_h^2). \tag{11}$$

For partial quantisation on the branches the Hilbert spaces are taken to be $\mathcal{H}(\mathbb{E}^-)$ and $\mathcal{H}(\mathbb{E}_h^2)$. The Hilbert space for the whole system in composite quantisation is given by $\mathcal{H}^c$ which can be decomposed as

$$\mathcal{H}^c = \oplus_n \mathcal{H}_n^c, \qquad \mathcal{H}_n^c = \mathcal{H}(\mathbb{E}^-) \oplus \mathcal{H}_n(\mathbb{E}_h^2). \tag{12}$$

2.2 Angular and Linear Momentum Operators

2.2.1 Partial and Composition Quantisation

The angular momentum operator in $\mathcal{H}(\mathbb{E}_h^2)$ is taken to be

$$\widehat{L}(\mathbb{E}_h^2) = \widehat{\mathbb{I}}(\mathbb{R}^+) \otimes \widehat{L}(\mathcal{C}). \tag{13}$$

This represents a property of our circuit which is totally localized in one branch of the circuit, i.e., in the plane, since there can be no angular momentum attached to the half-line. In other words the angular momentum operator in $\mathcal{H}(\mathbb{E}^-)$ must be taken as $\widehat{0}(\mathbb{E}^-)$. The composite angular momentum operator in $\mathcal{H}^c$ for the circuit as a whole is therefore

$$\widehat{L}^c = \widehat{0}(\mathbb{E}^-) \oplus \Big(\widehat{\mathbb{I}}(\mathbb{R}^+) \otimes \widehat{L}(\mathcal{C})\Big). \tag{14}$$

A wave function in $\mathcal{H}^c$ of the form

$$\eta_n^c = 0(\mathbb{E}^-) \oplus \Big(\phi_2(r) \otimes \eta_n(\theta)\Big) \tag{15}$$

is an eigenfunction of $\widehat{L}^c$ which can then be interpreted in the standard fashion as representing a state having angular momentum $n\hbar$. This is an example of local observables [7] (pp. 185–195).

Next, we can construct the linear momentum of the system from the linear momentum $p(\mathbb{E}^-)$ on the half-line $\mathbb{E}^-$ and the radial momentum $p_r(\mathbb{E}_h^2)$ on the plane $\mathbb{E}_h^2$. In partial quantisation the quantised versions of these two momenta are given by the following operators:

$$\widehat{p}_0(\mathbb{E}^-) = -i\hbar\left(\frac{d}{dx}\right)_0, \tag{16}$$

$$\widehat{p}_{r0}(\mathbb{E}_h^2) = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)_0 \otimes \widehat{\mathbb{I}}(\mathcal{C}). \tag{17}$$

The subscript 0 is used to indicate that the above operator expressions act respectively on the domains $C_0^\infty(\mathbb{E}^-)$ and $C_0^\infty(\mathbb{E}_h^2)$. These operators are not essentially selfadjoint, but only essentially strictly maximal symmetric [7] (p. 100, p. 127, p. 574), [11]. In composite quantisation we start with the direct sum

$$\widehat{P}_0^c = \widehat{p}_0(\mathbb{E}^-) \oplus \widehat{p}_{r0}(\mathbb{E}_h^2) \tag{18}$$

acting on the domain $\mathcal{D}(\widehat{P}_0^c) = \{\Psi_0^c \in C_0^\infty(\mathbb{E}^-) \oplus C_0^\infty(\mathbb{E}_h^2)\}$ in the composite Hilbert space $\mathcal{H}^c$. The operator $\widehat{P}_0^c$ is not selfadjoint, but has many selfadjoint extensions. For later comparison, let us consider the restriction $\widehat{P}_{0,n}^c$ of $\widehat{P}_0^c$ to $\mathcal{H}_n^c$, the subspace of $\mathcal{H}^c$ corresponding to angular momentum value $n\hbar$. Operator $\widehat{P}_{0,n}^c$ has deficiency indices (1,1) and hence admits a family of selfadjoint extensions $\widehat{P}_{\lambda_n}^c$ in $\mathcal{H}_n^c$ which can be characterized by a real parameter $\lambda_n \in (-\pi, \pi]$. An explicit construction of the family of selfadjoint extensions is detailed in Appendix 1. $\widehat{P}_{\lambda_n}^c$ possesses the following generalized eigenfunctions

$$\Psi_{\lambda_n,p}^c = e^{ipx} \oplus e^{i\lambda_n}\left(\frac{1}{\sqrt{2\pi r}} e^{ipr} \otimes \eta_n(\theta)\right), \quad p \neq 0,\ i = i/\hbar. \tag{19}$$

This wave function would represents a state with non-vanishing linear and angular momentum.

2.2.2 The Zero Angular Momentum Case

Let us consider the case when $n = 0$, i.e., consider various quantities in the subspace $\mathcal{H}_0^c$. Consistent with standard quantum theory we have the following interpretation:

1. The momentum operator in $\mathcal{H}_0^c$ is $\widehat{P}_{\lambda_0}^c$. Its eigenfunction $\Psi_{\lambda_0,p}^c$ represents the state of a beam of Cooper pairs travelling down the half-line with linear momentum $p \in (-\infty, \infty)$.
2. The probability current flowing down the line may be taken as p/m_c, where m_c is the mass of a Cooper pair, being twice that of an electron mass m_e [7] (p. 447, p. 535).
3. The probability current density in the plane is taken as $p/2\pi m_c r$ so that the total probability current flowing radially outwards is equal to p/m_c. The probability current is therefore conserved during this process since the total probability current radiating out at any radius on the plane is equal to the probability current flowing down the half-line into the plane.
4. There is no circulating probability current in the plane due to the absence of angular momentum.

5. The flow of Cooper pairs gives rise to an electric current, i.e., a supercurrent. This supercurrent is obtained by multiplying the charge of a Cooper pair q_c into the probability current. Here q_c, taken to be twice the elementary charge e, is positive.

2.3 Supercurrent Operator

Previous studies show that we can introduce a supercurrent operator, based on the linear momentum operator of the system [2] to [7]. In the subspace $\mathcal{H}_0^c$ the supercurrent operator is

$$\widehat{J}^c_{\lambda_0} = -\frac{q_c}{m_c}\widehat{P}^c_{\lambda_0} \tag{20}$$

which would admits $\Psi^c_{\lambda_0,p}$ as eigenfunction with eigenvalues $j = -(q_c/m_c)p$. In line with previous work we can use $\Psi^c_{\lambda_0,p}$ to represent a superconducting state with supercurrent j flowing down the half-line into the plane on which the current flows radially outwards without any circulating current on the plane. The wave function $\Psi^c_{\lambda_n,p}$ would entail a circulating supercurrent as well.

The linear momentum and the supercurrent operators may be called global observables which relate to wave functions on all the branches. In contrast, the angular momentum is a local observable localized in only one branch. When the system is in an angular momentum eigenstate η^c_n there is a circulating current on the plane round the origin but there is no current flowing down the half-line and no current flowing radially outwards on the plane.

To maintain a stable dc supercurrent we require the superconducting state to be an eigenfunction of the Hamiltonian of the system [7] (p. 519). In the next section we shall investigate whether there exists such a Hamiltonian.

2.4 Hamiltonians

For partial quantisation on the branches we have

$$\widehat{H}_{10} = -\frac{\hbar^2}{2m_c}\left(\frac{d^2}{dx^2}\right)_0 \quad \text{in } \mathcal{H}(\mathbb{E}^-), \tag{21}$$

$$\widehat{H}_{20} = -\frac{\hbar^2}{2m_c}\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)_0 \otimes \widehat{\mathbb{I}}(\mathcal{C}) + \left(\frac{1}{r^2}\right)_0 \widehat{\mathbb{I}}(\mathbb{R}^+) \otimes \frac{-1}{\hbar^2}\widehat{L}(\mathcal{C})^2\right\} \quad \text{in } \mathcal{H}(\mathbb{E}^2_h), \tag{22}$$

where the operator expressions

$$\left(\frac{d^2}{dx^2}\right)_0, \quad \text{and} \quad \left(\frac{1}{r^2}\right)_0, \quad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)_0 \tag{23}$$

are defined respectively on

$$C_0^\infty(\mathbb{E}^-) \quad \text{and} \quad \mathrm{C}_0^\infty(\mathbb{R}^+). \tag{24}$$

For composite quantisation for the circuit as a whole we first construct the following direct sum

$$\widehat{H}^c_0 = \widehat{H}_{10} \oplus \widehat{H}_{20} \quad \text{in } \mathcal{H}^c. \tag{25}$$

This operator has been studied in details [8]. It has deficiency indices (2,2) and hence admits a 4-parameter family of selfadjoint extensions specifiable by boundary conditions set out in Appendix 2. The composite Hamiltonian is to be chosen from one of these extensions.

2.5 Correlative Quantisation

We want to correlate the Hamiltonian with the supercurrent operator to achieve a state of stable dc supercurrent. Unfortunately $\Psi^c_{\lambda_n,p}$ violates the boundary conditions for any selfadjoint extension of $\widehat{H}^c_0$ (see Appendix 2). One can gain an appreciation of this result by showing that $\widehat{H}^c_0\Psi^c_{\lambda_n,p}$ is not even formally proportional to $\Psi^c_{\lambda_n,p}$. We will say that the Hamiltonian is incompatible with the supercurrent (linear momentum) operator. This implies that it is not possible to main a stable dc suppercurrent flowing down the half-line and radially outwards in the plane if the Hamiltonian for such a circuit is chosen from one of the selfadjoint extensions of H^c_0. This result is counter intuitive as one would think it physically possible to establish a stead dc supercurrent flowing from the half-line to the plane.

There is a way out of this difficulty: if the quantum theory on the plane is formulated in the *path space*, instead of the traditional physical space, we can find a Hamiltonian compatible with the supercurrent (linear momentum) operator.

3 Quantisation on the Path Space of $\mathbb{E}^2_h$

3.1 Path Space and Hilbert Space of Functions on the Path Space of $\mathbb{E}^2_h$

Equation (5) shows the construction of $L^2(\mathbb{E}^2_h)$ in terms of the tensor product of $L^2(\mathbb{R}^+, rdr)$ and $L^2(\mathcal{C}, d\theta)$. The latter is formed by functions on the circle $\mathcal{C}$. So far we have adopted the traditional coordinate representation of wave functions in the construction of Hilbert spaces. Such functions are defined on the physical space. As early as 1931 Dirac had pointed out that the position probability density function only determines the wave function up to an arbitrary phase [12]. Dirac suggested that it might be possible to generalize quantum mechanics in terms of wave functions defined on paths connecting points in the physical space, rather than wave functions defined directly on the physical space. In other words one should introduce the notion of *path spaces*, as opposed to physical spaces, and then define functions on these path spaces. While remaining single-valued in the path space these new wave functions may be multi-valued on the physical space. There have been many well-established path-space formulations of quantum mechanics [12–15]. We shall briefly set out a formulation directly applicable to our circuit configuration.

Let the physical space be denoted by $\mathbb{M}$. A path in $\mathbb{M}$ is a differentiable curve in $\mathbb{M}$. To construct a path space and functions on it we proceed as follows:

1. Choose a convenient point m_0 in $\mathbb{M}$ as a base point at the outset.
2. Link up any arbitrary point $m \in \mathbb{M}$ to m_0 by a differentiable curve (path) σ_m. Let $\Pi_m(\mathbb{M})$ denotes all the paths linking the point m to the fixed base point m_0. A path space $\Pi(\mathbb{M})$ of $\mathbb{M}$ is then formed by all these paths for all the point in $\mathbb{M}$, i.e.,
$$\Pi(\mathbb{M}) = \{\Pi_m(\mathbb{M}) : m \in \mathbb{M}\}. \tag{26}$$
Given m the paths in $\Pi_m(\mathbb{M})$ may or may not be homotopic, depending on the topological nature of $\mathbb{M}$.
3. One may impose various conditions to restrict the vast number of paths, even for each point m. One common condition is to regard all homotopic paths linking m to m_0 as equivalent. There may be many inequivalent paths for each point m, depending on the topology of $\mathbb{M}$.

4. One then defines functions $\Psi(\sigma_m)$ on the path space $\Pi(\mathbb{M})$ by mapping $\Pi(\mathbb{M})$ to the set of complex numbers $\mathbb{C}$. Again various conditions may be imposed on these functions. An obvious one would be to require its absolute value square $|\Psi(\sigma_m)|^2$ be single-valued for each $m \in \mathbb{M}$ independent of any particular path σ_m, since this is going to be related to a probability density function on $\mathbb{M}$ in quantum mechanics.
5. By further restriction to square-integrability we can arrive at a Hilbert space $\mathcal{H}(\Pi(\mathbb{M}))$ of functions on $\Pi(\mathbb{M})$. Such a construction is generally not unique, depending on the restrictions on both the nature of the paths and of the functions, and the topology of the physical space.

A description of an explicit formulation of such an approach for quantum particles confined to $\mathbb{E}_h^2$ is given by Wan, Bradshaw and Trueman [5, 6] and Wan [7] (pp. 637–657). Generally if the physical space is the Euclidean plane $\mathbb{E}^2$ then all the paths σ_m linking a given point m to m_0 are known to be homotopic and hence equivalent. As a result functions on path space would lead to the usual Hilbert space $L^2(\mathbb{E}^2)$ constructed from square-integrable functions on the physical space $\mathbb{E}^2$, and one would arrive at the same results as the traditional quantum theory set up directly $L^2(\mathbb{E}^2)$. However, $\mathbb{E}_h^2$ is topologically different from $\mathbb{E}^2$ and not all the paths linking m to m_0 are homotopic. Functions on the path space of $\mathbb{E}_h^2$ would give rise to new Hilbert spaces formed by wave functions which are multi-valued in $\mathbb{E}_h^2$. For our present application we need to establish a path space on the circle $\mathcal{C}$ in the plane, centered at the coordinate origin of the plane. A point on the circle is specified by the value of the polar angle θ. There is more to polar angle than meet the eye [7] (pp. 56–61, pp. 650–651). Allowing θ to be in the range $[0, 2\pi]$ the two extreme values $\theta = 0, 2\pi$ would refer to the same point in $\mathcal{C}$. Now, choose the fixed base point m_0 to be the point $\theta = 0$. As we move away from this base point anti-clockwise to another point m on the circle we trace out a path (curve) on the circle linking m_0 to m. Note that a curve or a path is a mapping from an interval of the reals to the circle. The mappings for moving clockwise and anti-clockwise are different, giving rise to different paths. There are infinite number of different paths from the base point m_0 to any point m depending on how many times the curve goes round the circle before ending up at the point m. These paths can be classified by an integer ℓ known as the winding number of the path. For examples we can have

1. a path, denoted by $\sigma_{m,0}$, from m_0 directly to m going anti-clockwise but without going round the circle,
2. a path, denoted by $\sigma_{m,1}$, from m_0 to m after going round the circle anti-clockwise once,
3. a path, denoted by $\sigma_{m,2}$, from m_0 to m after going round the circle anti-clockwise 2 times,
4. a path, denoted by $\sigma_{m,-1}$, from m_0 directly to m going clockwise but without going round the circle once,
5. a path, denoted by $\sigma_{m,-2}$, from m_0 to m after going round the circle clockwise 2 times.

Generally we shall denote a path by $\sigma_{m,\ell}$, where $m \in \mathcal{C}$ and $\ell = 0, \pm 1, \pm 2, \dots$. Let all the paths linking m_0 to m be denoted by $\Pi_m(\mathcal{C})$, and let

$$\Pi(\mathcal{C}) = \{\Pi_m(\mathcal{C}) : m \in \mathcal{C}\}. \tag{27}$$

We call $\Pi(\mathcal{C})$ the path space on the circle. Next, define functions on the path space by mappings

$$F : \Pi(\mathcal{C}) \to \mathbb{C} \quad \text{by} \quad F : \sigma_{m,\ell} \to F(\sigma_{m,\ell}) \in \mathbb{C}. \tag{28}$$

Then such functions are functions of both the point m and the winding number ℓ, i.e., $F = F(m, \ell)$, and hence are multi-valued on $\mathcal{C}$. The next question to settle is the nature of this multi-valuedness. To make this precise we can introduce an extended polar coordinate θ_{ex} which varies from $-\infty$ to ∞. The point is to use θ_{ex} to follow the point m as it goes round the circle in a given path. In other words, the value of θ_{ex} will specify both the position m on the circle as well as the winding number ℓ of the path. This is achieved by relating θ_{ex} to θ and the winding number ℓ by [7] (pp. 650–651).

$$\theta_{ex} = \theta + 2\ell\pi, \quad \theta \in [0, 2\pi). \tag{29}$$

Since we desire a probability interpretation of the function, i.e., $|F(m,\ell)|^2$ is to be interpreted as the position probability density of the particle on the circle, we should restrict functions to the form:

$$F = F_\gamma(m,\ell) = F_\gamma(m,\ell) = e^{i\ell\gamma} f(m), \tag{30}$$

where γ is a real constant in the range $[0, 2\pi)$. The main feature here is that $F_\gamma(m,\ell)$ is not a single-valued function of m and does not satisfy the periodic boundary condition in (7), unless γ is chosen to be zero. An alternative description is to rewrite the function as a single-valued function of θ_{ex}, denoted by $\eta_\gamma(\theta_{ex})$, i.e.,

$$\eta_\gamma(\theta_{ex}) = e^{i\ell\gamma} f(m). \tag{31}$$

Such a function satisfies the following quasi-periodic boundary condition:

$$\eta_\gamma(\theta_{ex} + 2\pi) = e^{i\gamma}\eta_\gamma(\theta_{ex}). \tag{32}$$

For a given γ the set of functions of this form which are square-integrable over the range of $\theta_{ex} \in [0, 2\pi]$, i.e.,

$$\int_0^{2\pi} \eta_\gamma(\theta_{ex})^* \eta_\gamma(\theta_{ex}) d\theta_{ex} < \infty, \tag{33}$$

forms a Hilbert space, to be denoted by $\mathcal{H}_\gamma(\Pi(\mathcal{C}))$. In this way we arrive at a one-parameter family of Hilbert spaces $\mathcal{H}_\gamma(\Pi(\mathcal{C}))$, $\gamma \in [0, 2\pi)$, based on functions $\eta_\gamma(\theta_{ex})$ on the path space $\Pi(\mathcal{C})$ of the circle [7] (p. 649).

By replacing $L^2(\mathcal{C}, d\theta)$ in (5) we obtain a new Hilbert space $\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2))$ constructed from the tensor product of $L^2(\mathbb{R}^+, rdr)$ and $\mathcal{H}_\gamma(\Pi(\mathcal{C}))$, i.e.,

$$\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2)) = L^2(\mathbb{R}^+, rdr) \otimes \mathcal{H}_\gamma(\Pi(\mathcal{C})). \tag{34}$$

Members of $\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2))$ are linear combinations of functions of the form $\phi_2(r) \otimes \eta_\gamma(\theta_{ex})$. The seemingly arbitrary parameter γ can be chosen by the physical properties of the system, as will be seen later.

3.2 Angular Momentum and Radial Momentum in $\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2))$

For the angular momentum in $\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2))$ we can adopt operator expression obtained by a slight modification of $\widehat{L}(\mathcal{C})$ introduced earlier, i.e., we have [7] (p. 654–655),

$$\widehat{L}_\gamma = \widehat{\mathbb{I}}(\mathbb{R}^+) \otimes \left(-i\hbar \frac{\partial}{\partial\theta_{ex}}\right). \tag{35}$$

This operator is selfadjoint and admits eigenfunctions of the form $\phi_2(r) \otimes \eta_{\gamma,n}(\theta_{ex})$ where

$$\eta_{\gamma,n}(\theta_{ex}) = e^{\dot{\imath} L_{\gamma,n}\theta_{ex}}, \quad n = 0, \pm 1, \pm 2, \dots, \tag{36}$$

and

$$L_{\gamma,n} = \hbar\left(n + \frac{\gamma}{2\pi}\right) \tag{37}$$

which are the corresponding eigenvalues. These eigenfunctions satisfy the quasi-periodic boundary condition in (32). A striking feature here is the absence of a zero angular momentum eigenstate, i.e., $L_{\gamma,n} \neq 0$ unless $\gamma = 0$. This will have important consequences in composite quantisation.

Each $\eta_{\gamma,n}$ generates a subspace $\mathcal{L}_{\gamma,n}(\Pi(\mathcal{C}))$ of $\mathcal{H}_\gamma(\Pi(\mathcal{C}))$ with $\mathcal{H}_\gamma(\Pi(\mathcal{C})) = \oplus_{\mathcal{C}n}\mathcal{L}_{\gamma,n}(\Pi(\mathbb{E}_h^2))$. To avoid confusion later we have inserted a subscript $\mathcal{C}$ to indicate a direct sum in $\mathcal{H}_\gamma(\Pi(\mathcal{C}))$. Let

$$\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2)) = L^2(\mathbb{R}^+, rdr) \otimes \mathcal{L}_{\gamma,n}(\Pi(\mathcal{C})). \tag{38}$$

Then we have

$$\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2)) = L^2(\mathbb{R}^+, rdr) \otimes \Big(\oplus_{\mathcal{C}n}\, \mathcal{L}_{\gamma,n}(\Pi(\mathcal{C}))\Big) \tag{39}$$

$$= \oplus_{\mathcal{C}n}\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2)). \tag{40}$$

As it will become apparent later it is the subspaces $\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2))$ we shall be working in, rather than $\mathcal{H}_\gamma(\Pi(\mathbb{E}_h^2))$. We shall denote the restriction of the angular momentum operator $\widehat{L}_\gamma$ to $\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2))$ by $\widehat{L}_{\gamma,n}$. For the radial momentum in $\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2))$ we have

$$\widehat{p}_{r0\gamma,n} = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)_0 \otimes \widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C})), \tag{41}$$

where $\widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C}))$ is the restriction of the identity operator $\widehat{\mathbb{I}}_\gamma(\Pi(\mathcal{C}))$ on $\mathcal{H}_\gamma(\Pi(\mathcal{C}))$ to $\mathcal{L}_{\gamma,n}(\Pi(\mathcal{C}))$.

3.3 Hamiltonian in $\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2))$

3.3.1 General Consideration

Following (22) the partially quantised Hamiltonian in $\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2))$ is defined by the following operator:

$$\widehat{H}_{0\gamma,n}(\Pi(\mathbb{E}_h^2)) = -\frac{\hbar^2}{2m_c}\left\{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)_0 \otimes \widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C})) + \left(\frac{1}{r^2}\right)_0 \widehat{\mathbb{I}}(\mathbb{R}^+) \otimes \frac{-1}{\hbar^2}\widehat{L}^2_{\gamma,n}\right\}, \tag{42}$$

which can be rewritten as

$$\widehat{H}_{0\gamma,n} = -\frac{\hbar^2}{2m_c}\left\{\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)_0 - \left(\frac{1}{r^2}\right)_0 \left(n + \frac{\gamma}{2\pi}\right)^2\right] \otimes \widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C}))\right\}. \tag{43}$$

There are two special cases crucial in correlative quantisation later:

1. Case 1 $\gamma=\pi$ and $n=0$. The Hamiltonian in $\mathcal{H}_{\pi,0}(\Pi(\mathbb{E}_h^2))$ reduces to

$$\widehat{H}_{0\pi,0}=-\frac{\hbar^2}{2m_c}\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{4r^2}\right)_0\otimes\widehat{\mathbb{I}}_{\pi,0}(\Pi(\mathcal{C}))\right\}. \tag{44}$$

2. Case 2 $\gamma=\pi$ and $n=-1$. The Hamiltonian $\mathcal{H}_{\pi,-1}(\Pi(\mathbb{E}_h^2))$ reduces to

$$\widehat{H}_{0\pi,-1}=-\frac{\hbar^2}{2m_c}\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{4r^2}\right)_0\otimes\widehat{\mathbb{I}}_{\pi,-1}(\Pi(\mathcal{C}))\right\}. \tag{45}$$

Comparing with (41) and (37) we see that:

1. Case 1 $\gamma=\pi$ and $n=0$:
   (a) The quasi-periodic boundary condition in (32) reduces to $\eta_\gamma(\theta_{ex}+2\pi)=-\eta_\gamma(\theta_{ex})$.
   (b) The angular momentum operator is $\widehat{L}_{\pi,0}$ which has eigenvalue $L_{\pi,0}=\hbar/2$ in $\mathcal{H}_{\pi,0}(\Pi(\mathbb{E}_h^2))$.
   (c) The Hamiltonian becomes

$$\widehat{H}_{0\pi,0}=\frac{1}{2m_c}\left(\widehat{p}_{r0\pi,0}\right)^2. \tag{46}$$

2. Case 2 $\gamma=\pi$ and $n=-1$:
   (a) The quasi-periodic boundary condition in (32) reduces to $\eta_\gamma(\theta_{ex}+2\pi)=-\eta_\gamma(\theta_{ex})$.
   (b) The angular momentum operator is $\widehat{L}_{\pi,-1}$ which has eigenvalue $L_{\pi,-1}=-\hbar/2$ in $\mathcal{H}_{\pi,-1}(\Pi(\mathbb{E}_h^2))$.
   (c) The Hamiltonian reduces to

$$\widehat{H}_{0\pi,-1}=\frac{1}{2m_c}\left(\widehat{p}_{r0\pi,-1}\right)^2. \tag{47}$$

3.3.2 Notation: Subspaces and Operators in Them

Since we shall only be considering the cases with $\gamma=\pi$ and $n=0,-1$ we will introduce the following notation to highlight the angular momentum values:

1. The subspace $\mathcal{H}_{\pi,0}(\Pi(\mathbb{E}_h^2))$ with angular momentum value $+\hbar/2$ will be denoted by $\mathcal{H}_+(\Pi(\mathbb{E}_h^2))$. The operators $\widehat{p}_{r0\pi,0}$, $\widehat{L}_{\pi,0}$ and $\widehat{H}_{0\pi,0}$ will be denoted by $\widehat{p}_{r0,+}$, $\widehat{L}_+$ and $\widehat{H}_{0,+}$ respectively.
2. The subspace $\mathcal{H}_{\pi,-1}(\Pi(\mathbb{E}_h^2))$ with angular momentum value $-\hbar/2$ will be denoted by $\mathcal{H}_-(\Pi(\mathbb{E}_h^2))$. The operators $\widehat{p}_{r0\pi,-1}$, $\widehat{L}_{\pi,-1}$ and $\widehat{H}_{0\pi,-1}$ will be denoted by $\widehat{p}_{r0,-}$, $\widehat{L}_-$ and $\widehat{H}_{0,-}$ respectively.
3. The eigenfunctions of $\widehat{L}_+$ and $\widehat{L}_-$ are similarly denoted, i.e., $\eta_+=\eta_{\pi,0}$ and $\eta_-=\eta_{\pi,-1}$.

3.3.3 Notation: Direct Sums of Subspaces and Operators

We now introduce the following direct sums:

1. The direct sum of the spaces $\mathcal{H}_+(\Pi(\mathbb{E}_h^2))$ and $\mathcal{H}_-(\Pi(\mathbb{E}_h^2))$ is denoted by $\mathcal{H}_\pm(\Pi(\mathbb{E}_h^2))$, i.e.,

$$\mathcal{H}_\pm(\Pi(\mathbb{E}_h^2))=\mathcal{H}_+(\Pi(\mathbb{E}_h^2))\oplus_{\mathcal{C}}\mathcal{H}_{-1}(\Pi(\mathbb{E}_h^2)) \tag{48}$$

$$= L^2(\mathbb{R}^+, rdr) \otimes \Big(\mathcal{L}_{\pi,0}(\Pi(\mathcal{C})) \oplus_{\mathcal{C}} \mathcal{L}_{\pi,-1}(\Pi(\mathcal{C}))\Big). \tag{49}$$

The composite subscript $\pm$ is used to indicate that the direct sum space corresponds to angular momentum values $\pm\hbar/2$.

2. The direct sum of angular momentum operators $\widehat{L}_+$ and $\widehat{L}_-$:

$$\widehat{L}_\pm = \widehat{L}_+ \oplus_{\mathcal{C}} \widehat{L}_-. \tag{50}$$

3. The direct sum of radial momentum operators $\widehat{p}_{r0,+}$ and $\widehat{p}_{r0,-}$:

$$\widehat{p}_{r0,\pm} = \widehat{p}_{r0,+} \oplus_{\mathcal{C}} \widehat{p}_{r0,-}. \tag{51}$$

4. The direct sum of the Hamiltonians $\widehat{H}_{0,+}$ and $\widehat{H}_{0,-}$:

$$\widehat{H}_{0,\pm} = \widehat{H}_{0,+} \oplus_{\mathcal{C}} \widehat{H}_{0,-} = \frac{1}{2m_c}\Big(\big(\widehat{p}_{r0,+}\big)^2 \oplus_{\mathcal{C}} \big(\widehat{p}_{r0,-}\big)^2\Big) = \frac{1}{2m_c}\big(\widehat{p}_{r0,\pm}\big)^2. \tag{52}$$

4 Quantisation on the Path Space of $\mathbb{E}^- \times \mathbb{E}_h^2$

4.1 The Hilbert Space

For the circuit geometry $\mathbb{E}^- \times \mathbb{E}_h^2$ we can carry out composite quantisation incorporating the path space on the circle by introducing the following Hilbert spaces

$$\mathcal{H}^c_{\gamma,n} = L^2(\mathbb{E}^-) \oplus \mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2)), \tag{53}$$

$$\mathcal{H}^c_{\gamma} = L^2(\mathbb{E}^-) \oplus \mathcal{H}_{\gamma}(\Pi(\mathbb{E}_h^2)), \tag{54}$$

$$= L^2(\mathbb{E}^-) \oplus \big(\oplus_{\mathcal{C}n} \mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}_h^2))\big), \tag{55}$$

$$= \oplus_{\mathcal{C}n} \mathcal{H}^c_{\gamma,n}. \tag{56}$$

At first sight $\mathcal{H}^c_\gamma$ should be the Hilbert space associated with the circuit geometry $\mathbb{E}^- \times \mathbb{E}_h^2$. As it turns out and contrary to expectation, it is $\mathcal{H}^c_{\gamma,n}$ which is directly associated with the superconducting circuit system, on account of the compatibility requirement in correlative quantisation. So, having partially quantised various quantities already we shall commence the composite quantisation process by defining operators in $\mathcal{H}^c_{\gamma,n}$.

4.2 Angular and Linear Momentum Operators

First we shall define the angular momentum operator $\widehat{L}^c_{\gamma,n}$ in $\mathcal{H}^c_{\gamma,n}$ by

$$\widehat{L}^c_{\gamma,n} = \widehat{0}(\mathbb{E}^-) \oplus \widehat{L}_{\gamma,n}. \tag{57}$$

For linear momentum in $\mathcal{H}^c_{\gamma,n}$ we define the following operator

$$\widehat{P}^c_{0\gamma,n} = -i\hbar\left(\frac{d}{dx}\right)_0 \oplus \left(-i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)_0 \otimes \widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C}))\right). \tag{58}$$

This operator has a family of selfadjoint extensions $\widehat{P}^c_{\gamma,\lambda_n}$ in $\mathcal{H}^c_{\gamma,n}$, which is parameterizabled by $\lambda_n \in (-\pi,\pi]$ (see Appendix 3) and admits generalized eigenfunctions of the form

$$\Psi^c_{\gamma,\lambda_n,p} = e^{\dot{\imath}px} \oplus e^{i\lambda_n}\psi_{p,\gamma,n}(r,\theta_{ex}), \tag{59}$$

where

$$\psi_{\gamma,n,p}(r,\theta_{ex}) = \frac{1}{\sqrt{2\pi r}} e^{\dot{\imath}pr} \otimes \eta_{\gamma,n}(\theta_{ex}). \tag{60}$$

These wave functions entail a non-zero angular momentum, unless $\gamma = 0$ and $n = 0$. The linear momentum of the system shall be determined from the set of selfadjoint extensions $\widehat{P}^c_{\gamma,\lambda_n}$ of $\widehat{P}^c_{0\gamma,n}$ in conjunction with the Hamiltonian in correlative quantisation later. As before we shall adopt the following the notation for $\widehat{L}^c_{\gamma,n}$, $\widehat{P}^c_{0\gamma,n}$, $\widehat{P}^c_{\gamma,\lambda_n}$, $\eta_{\gamma,n}$, $\psi_{\gamma,n,p}$, $\Psi^c_{\gamma,\lambda_n,p}$ and $\mathcal{H}^c_{\gamma,n}$:

1. Case 1 $\quad \gamma = \pi$ and $n = 0$. We have $\widehat{L}^c_+$, $\widehat{P}^c_{0,+}$, $\widehat{P}^c_{\lambda_+}$, η_+, $\psi_{+,p}$, $\Psi^c_{\lambda_+,p}$, $\mathcal{H}^c_+$.
2. Case 2 $\quad \gamma = \pi$ and $n = -1$. We have $\widehat{L}^c_-$, $\widehat{P}^c_{0,-}$, $\widehat{P}^c_{\lambda_-}$, η_-, $\psi_{-,p}$, $\Psi^c_{\lambda_-,p}$, $\mathcal{H}^c_-$.

4.3 Hamiltonians

Similar to (25) the composite Hamiltonian of the system in $\mathcal{H}^c_{\gamma,n}$ is obtained from an appropriate selfadjoint extension of

$$\widehat{H}^c_{0\gamma,n} = -\frac{\hbar^2}{2m_c}\left\{\left(\frac{d^2}{dx^2}\right)_0 \oplus \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)_0 \otimes \widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C})) + \left(\frac{1}{r^2}\right)_0 \widehat{\mathbb{I}}(\mathbb{R}^+) \otimes \frac{-1}{\hbar^2}\widehat{L}^2_{\gamma,n}\right]\right\}. \tag{61}$$

This operator can be rewritten as

$$\widehat{H}^c_{0\gamma,n} = -\frac{\hbar^2}{2m_c}\left\{\left(\frac{d^2}{dx^2}\right)_0 \oplus \left[\left(\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\left(n + \frac{\gamma}{2\pi}\right)^2\right)_0 \otimes \widehat{\mathbb{I}}_{\gamma,n}(\Pi(\mathcal{C}))\right]\right\}. \tag{62}$$

Again there are two special cases:

1. Case 1 $\quad \gamma = \pi$ and $n = 0$. The operator $\widehat{H}^c_{0,+}$ in $\mathcal{H}^c_+$ reduces to

$$\widehat{H}^c_{0,+} = \frac{1}{2m_c}\left(\widehat{P}^c_{0,+}\right)^2. \tag{63}$$

We can immediately identify a family of selfadjoint extensions

$$\widehat{H}^c_{\lambda_+} = \frac{1}{2m_c}\left(\widehat{P}^c_{\lambda_+}\right)^2. \tag{64}$$

Clearly this Hamiltonian is compatible with $\widehat{P}^c_{\lambda_+}$ and $\widehat{L}^c_+$; it admits eigenfunctions of the form $\Psi^c_{\lambda_+,p}$ corresponding to linear momentum, angular momentum and energy eigenvalues p, $\hbar/2$ and $E_+ = E = p^2/2m_c$ respectively.

2. Case 2 $\gamma = \pi$ and $n = -1$. The operator $\widehat{H}^c_{0,-}$ in $\mathcal{H}^c_-$, reduces to

$$\widehat{H}^c_{0,-} = \frac{1}{2m_c}\left(\widehat{P}^c_{0,-}\right)^2. \tag{65}$$

We can immediately identify a family of selfadjoint extensions

$$\widehat{H}^c_{\lambda_-} = \frac{1}{2m_c}\left(\widehat{P}^c_{\lambda_-}\right)^2. \tag{66}$$

This Hamiltonian is compatible with $\widehat{P}^c_-$ and $\widehat{L}^c_-$; it admits eigenfunctions of the form $\Psi^c_{\lambda_-,p}$ corresponding to linear momentum, angular momentum and energy eigenvalues p, $-\hbar/2$ and $E_- = E = p^2/2m_c$ respectively.

As before we can define the following direct sums:

1. The direct sum space

$$\mathcal{H}^c_\pm = \mathcal{H}^c_+ \oplus_C \mathcal{H}^c_-. \tag{67}$$

2. The direct sum angular momentum operator

$$\widehat{L}^c_\pm = \widehat{L}^c_+ \oplus_C \widehat{L}^c_-. \tag{68}$$

3. The direct sum momentum operator $\widehat{P}^c_\pm = \widehat{P}^c_{\lambda_+} \oplus_C \widehat{P}^c_{\lambda_-}$, where we have chosen a common parameter $\lambda_\pm$, i.e., we set

$$\lambda_+ = \lambda_- = \lambda_\pm. \tag{69}$$

Generalized eigenfunctions of $\widehat{P}^c_\pm$ are of the form

$$\Psi^c_{\lambda_\pm,p,(a_+,a_-)} = e^{\dot{\imath}px} \oplus e^{i\lambda_\pm}\left(\frac{1}{\sqrt{2\pi r}} e^{\dot{\imath}pr} \otimes \eta_{a_+,a_-}(\theta_{ex})\right), \tag{70}$$

where

$$\eta_{a_+,a_-}(\theta_{ex}) = a_+\eta_+(\theta_{ex}) + a_-\eta_-(\theta_{ex}). \tag{71}$$

It is also possible to construct a direct sum with different λ_+ and λ_-. Although we have chosen $\lambda_+ = \lambda_- = \lambda_\pm$ we shall still retain their separate identity in what follows to avoid confusion.

4. A corresponding Hamiltonian in $\mathcal{H}^c_\pm$ is

$$\widehat{H}^c_\pm = \frac{1}{2m}\left(\widehat{P}^c_\pm\right)^2, \tag{72}$$

which shares the generalized eigenfunctions of $\widehat{P}^c_\pm$.

4.4 Correlative Quantisation

The results in the preceding section show that there exist compatible momentum and Hamiltonian operators. The compatibility requirement on the Hamiltonian and the momentum for our superconducting circuit serves to select a single value π for γ and two possible values, $\pm\hbar/2$, for the angular momentum of the system. We have now arrived at three possible prescriptions:

1. Choose the Hilbert space $\mathcal{H}_{+}^{c}$. Then the linear and angular momenta are represented by $\widehat{P}_{\lambda_+}^{c}$ and $\widehat{L}_{+}^{c}$. The Hamiltonian corresponds to $\widehat{H}_{+}^{c}$. These three observables are compatible and share eigenfunctions of the form $\Psi_{\lambda_+,p}^{c}$.
2. Choose the Hilbert space $\mathcal{H}_{-}^{c}$. Then the linear and angular momenta are represented by $\widehat{P}_{\lambda_-}^{c}$ and $\widehat{L}_{-}^{c}$. The Hamiltonian corresponds to $\widehat{H}_{-}^{c}$. These three observables are compatible and share eigenfunctions of the form $\Psi_{\lambda_-,p}^{c}$.
3. Choose the Hilbert space $\mathcal{H}_{\pm}^{c}$. Then the linear momentum is represented by $\widehat{P}_{\pm}^{c}$. The Hamiltonian corresponds to $\widehat{H}_{\pm}^{c}$. These two observables are compatible and share eigenfunctions of the form $\Psi_{\lambda_\pm,p,(a_+,a_-)}^{c}$. However, the angular momentum operator $\widehat{L}_{\pm}^{c}$ does not admit $\Psi_{\lambda_\pm,p,(a_+,a_-)}^{c}$ as an eigenfunction unless either a_+ or a_- is zero.

The natural choice would be to associate the Hilbert space $\mathcal{H}_{\pm}^{c}$ with our circuit system with observables $\widehat{P}_{\pm}^{c}$, $\widehat{L}_{\pm}^{c}$ and $\widehat{H}_{\pm}^{c}$ and with the macroscopic wave function $\Psi_{\lambda_\pm,p,(a_+,a_-)}^{c}$ describing the state.

4.5 Interpretation

We have achieved our goal in deriving a theory which can describe a steady dc supercurrent $j = -(q_c/m_c)p$ flowing down the wire and then spreading out radially in the plane. A superconducting state is described by either $\Psi_{\lambda_+,p}^{c}$ or $\Psi_{\lambda_-,p}^{c}$, or a linear combination of them $\Psi_{\lambda_\pm,p,(a_+,a_-)}^{c}$.

Our theory also gives rise to an angular momentum. The fact that the angular momentum in $\Psi_{\lambda_+,p}^{c}$ and $\Psi_{\lambda_-,p}^{c}$ are not zero suggests the existence of an additional circulating supercurrent on the plane round the origin. We can go on to define the magnetic moment or magnetic flux generated by the circulating current [7] (pp. 486–489), [16, 17]:

1. The circulating current is a local quantity localized in the plane. Only the component of the wave function on the plane, i.e., $\psi_{\pi,n,p}$ in (60), contributes directly to it. Since $\psi_{\pi,n,p}$ is an eigenfunction of $\widehat{L}_{\pi,n}$ corresponding to the eigenvalue $L_{\pi,n}$ a formal calculation will give the following circulating electric current density on the plane [7] (pp. 486–489), [16, 17]:

$$j_n(r) = -q_c \frac{L_{\pi,n}}{m_c r} \left|\psi_{\pi,n,p}\right|^2 = -q_c \frac{L_{\pi,n}}{2\pi m_c r^2}. \tag{73}$$

   The current circulating in an annulus about the origin of thickness dr is $dj_n(r) = j_n(r)dr$.
2. The magnetic moment associated with the current circulating in an annulus of thickness dr is

$$\pi r^2 dj_n(r) = \pi r^2 j_n(r)dr = -q_c \frac{L_{\pi,n}}{2m_c} dr. \tag{74}$$

   An attempt to integrate this to obtain the total magnetic moment fails because $\psi_{\pi,n,p}$ is a generalized eigenfunction of the radial momentum on the plane which is not normalizable. The normalization procedure set out in Appendix 4 has to be employed. The analysis in Appendix 4 shows that we should take the normalized magnetic moment element due to current in the annulus defined by radii $r_a < r_b$ to be

$$dM_n = \frac{\int_{r_a}^{r_b} \pi r^2 j_n(r)dr}{\int_{r_a}^{r_b}\int_0^{2\pi} |\psi_{\pi,n,p}|^2 r dr d\theta} \tag{75}$$

so that the total magnetic moment over the plane is

$$M_n = \lim_{r_a \to 0, r_b \to \infty} \frac{\int_{r_a}^{r_b} \pi r^2 j_n(r) dr}{\int_{r_a}^{r_b} \int_0^{2\pi} |\psi_{\pi,n,p}|^2 r dr d\theta_{ex}} = -\frac{q_c}{2m_c} L_{\pi,n}. \tag{76}$$

We can also define a corresponding magnetic moment operator in $\mathcal{H}^c_{\pi,n}$:

$$\widehat{M}_n = -\frac{q_c}{2m_c} \widehat{L}_{\pi,n}. \tag{77}$$

3. For the magnetic flux associated with the current circulating in an annulus of thickness dr we can make use of the expression for the self-inductance

$$L_{q_c}(r) = m_c \left(\frac{2\pi r}{q_c^2} \right)^2 \tag{78}$$

established for similar current configurations [7] (p. 483, p. 488), [17]. Note that we have used the expression L_{φ_n} in [17] for the self-inductance since the wave function $\psi_{\pi,n,p}$ contains the effective normalization factor $1/\sqrt{2\pi r}$ for circular motion. We get

$$d\Phi_n = \left(\frac{\int_{r_a}^{r_b} L_{q_c}(r) j_n(r) dr}{\int_{r_a}^{r_b} \int_0^{2\pi} |\psi_{\pi,n,p}|^2 r dr d\theta_{ex}} \right) \tag{79}$$

so that the total magnetic flux given rise by the circulating current over the plane is

$$\Phi_n = \lim_{r_a \to 0, r_a \to \infty} \frac{\int_{r_a}^{r_b} L_{q_c}(r) j_n(r) dr}{\int_{r_a}^{r_b} \int_0^{2\pi} |\psi_{\pi,n,p}|^2 r dr d\theta_{ex}} \tag{80}$$

$$= -\frac{2\pi}{q_c} L_{\pi,n}. \tag{81}$$

We can also define a corresponding flux operator

$$\widehat{\Phi}_n = -\frac{2\pi}{q_c} \widehat{L}_{\pi,n}. \tag{82}$$

4. For the two cases of interest, i.e., when $n = 0, -1$, we have:
   (a) The magnetic moment:

$$\begin{array}{ll} M_+ = -\dfrac{q_c \hbar}{4m_c} & \text{for } n = 0 \text{ with the corresponding operator denoted by } \widehat{M}_+ \\ M_- = \dfrac{q_c \hbar}{4m_c} & \text{for } n = -1 \text{ with the corresponding operator denoted by } \widehat{M}_- \end{array}. \tag{83}$$

   In terms of Bohr magneton μ_B for an orbiting electron in an atom we have [18] (p. 160)

$$M_+ = -\frac{1}{2}\mu_B, \quad M_- = \frac{1}{2}\mu_B, \quad \mu_B = \frac{e\hbar}{2m_e}. \tag{84}$$

(b) Magnetic flux:

$$\begin{array}{ll}\Phi_+ = -\frac{1}{2}\Phi_c & \text{for } n=0 \text{ with corresponding operator denoted by } \widehat{\Phi}_+ \\ \Phi_- = +\frac{1}{2}\Phi_c & \text{for } n=-1 \text{ with corresponding operator denoted by } \widehat{\Phi}_-\end{array}, \tag{85}$$

where $\Phi_c = h/q_c$ is the Cooper pair magnetic flux quantum associated with a superconducting ring [7] (p. 515). Our present values are half that of the Cooper pair magnetic flux quantum.

5. Physically the wave function $\Psi^c_{\lambda_+,p}$ describes a steady dc superconducting state in which a steady dc supercurrent $j_c = -(q_c/m_c)p$ flows down the wire into the plane. On the plane there is supercurrent flowing radially outwards with a total current outflow equal to j_c. In addition there is a circulating current in the plane generating a magnetic moment M_+ and a magnetic flux Φ_+. Similar remarks apply to $\Psi^c_{\lambda_-,p}$.
6. In the direct sum space $\mathcal{H}_\pm(\Pi(\mathbb{E}^2_h))$ we can define the magnetic moment and magnetic flux operators as

$$\widehat{M}_\pm = \widehat{M}_+ \oplus_{\mathcal{C}} \widehat{M}_-, \tag{86}$$

$$\widehat{\Phi}_\pm = \widehat{\Phi}_+ \oplus_{\mathcal{C}} \widehat{\Phi}_-. \tag{87}$$

7. All the above operators defined in $\mathcal{H}_+(\Pi(\mathbb{E}^2_h))$, $\mathcal{H}_-(\Pi(\mathbb{E}^2_h))$ and $\mathcal{H}_\pm(\Pi(\mathbb{E}^2_h))$ can be extended to the composite Hilbert spaces $\mathcal{H}^c_+$, $\mathcal{H}^c_-$ and $\mathcal{H}^c_\pm$, i.e., we have

$$\widehat{M}^c_+ = \widehat{0}(\mathbb{E}^-) \oplus \widehat{M}_+,\ \widehat{M}^c_- = \widehat{0}(\mathbb{E}^-) \oplus \widehat{M}_-,\ \widehat{M}^c_\pm = \widehat{0}(\mathbb{E}^-) \oplus \widehat{M}_\pm, \tag{88}$$

and similar for $\widehat{\Phi}^c_+$, $\widehat{\Phi}^c_-$ and $\widehat{\Phi}^c_\pm$.
8. In the composite space $\mathcal{H}^c_\pm$ there is the possibility of a superposition of a pair of opposing circulating supercurrents in state $\Psi^c_{\lambda_\pm,p,(a_+,a_-)}$. For examples for $a_+ = 1/\sqrt{2}$ and $a_- = \pm 1/\sqrt{2}$ we have the following linear combinations:

$$\Psi^c_{\lambda_\pm,p,(+)} = \frac{1}{\sqrt{2}}\left(\Psi^c_{\lambda_+,p} + \Psi^c_{\lambda_-,p}\right) = e^{\dot{\imath}px} \oplus \frac{e^{i\lambda_\pm}}{\sqrt{2}}\left(\psi_{+,p}(r,\theta_{ex}) + \psi_{-,p}(r,\theta_{ex})\right), \tag{89}$$

$$\Psi^c_{\lambda_\pm,p,(-)} = \frac{1}{\sqrt{2}}\left(\Psi^c_{\lambda_+,p} - \Psi^c_{\lambda_-,p}\right) = e^{\dot{\imath}px} \oplus \frac{e^{i\lambda_\pm}}{\sqrt{2}}\left(\psi_{+,p}(r,\theta_{ex}) - \psi_{-,p}(r,\theta_{ex})\right). \tag{90}$$

These two combinations correspond to a zero angular momentum expectation value, and hence a zero current expectation value, e.g.,

$$\langle \Psi^c_{\lambda_\pm,p,(+)} \mid \widehat{L}^c_\pm \Psi^c_{\lambda_\pm,p,(+)} \rangle \tag{91}$$

$$= \lim_{r_a\to 0, r_a\to\infty} \frac{\int_{r_a}^{r_b}\int_0^{2\pi} (\psi_{+,p}+\psi_{-,p})^* \widehat{L}_\pm (\psi_{+,p}+\psi_{-,p}) r dr d\theta_{ex}}{\int_{r_a}^{r_b}\int_0^{2\pi} |(\psi_{+,p}+\psi_{-,p})|^2 r dr d\theta_{ex}} = 0. \tag{92}$$

Intuitively this corresponds to having currents flowing in both clockwise and anticlockwise directions. The energy expectation value is $\langle \Psi^c_{\lambda_\pm,p,(+)} \mid \widehat{L}^c_\pm \Psi^c_{\lambda_\pm,p,(+)} \rangle = E = p^2/2m_c$.
9. As in the case of a superconducting ring with a Josephson junction a superposition of two opposing circular supercurrents can be regarded as a Schrödinger's cat state [19]. We have a similar situation here in $\Psi^c_{\lambda_\pm,p,(+)}$ which will be investigated in the next section.

5 Schrödinger's Cat States

One way to facilitate an experimental test on the existence of the various states discussed earlier is to separate them in terms of their energy, e.g., one may want to separate the two states $\Psi^c_{\lambda_+,p}$ and $\Psi^c_{\lambda_-,p}$ in terms of their energy [19]. This can be achieved by the application of an external magnetic field to induce a Zeeman effect in the condensate. A uniform and constant external magnetic field of magnitude B in the x-direction is applied perpendicularly to the superconducting plane. Let us consider the following three cases in the weak field approximation [16, 20]:

1. In $\mathcal{H}^c_+$ the magnetic field would amount to the addition to the Hamiltonian of a term $-B\widehat{M}^c_+$, i.e., the Hamiltonian $\widehat{H}^c_+$ becomes

$$\widehat{H}^c_{+,B} = \widehat{H}^c_+ - B\widehat{M}^c_+. \tag{93}$$

   This new Hamiltonian admits $\Psi^c_{\lambda_+,p}$ as an eigenfunction but with a higher energy value $E_{+,B}$ given by

$$E_{+,B} = E + \Delta E, \quad E = \frac{p^2}{2m_c}, \quad \Delta E = \frac{1}{2}B\mu_B. \tag{94}$$

2. Similar results apply to $\mathcal{H}^c_-$ leading to a new Hamiltonian $\widehat{H}^c_{-,B}$ which admits $\Psi^c_{\lambda_-,p}$ as an eigenfunction but with a lower energy value

$$E_{-,B} = E - \Delta E. \tag{95}$$

3. In the Hilbert space $\mathcal{H}^c_{\pi,\pm}$ the Hamiltonian is

$$\widehat{H}^c_{\pm,B} = \widehat{H}^c_\pm - B\widehat{M}^c_\pm. \tag{96}$$

   The wave function $\Psi^c_{\lambda_\pm,p,(+)}$ is not an eigenstate of this Hamiltonian. Under $\widehat{H}^c_{\pm,B}$ the wave function $\Psi^c_{\lambda_\pm,p,(+)}$ will evolve according to

$$\Psi^c_{\lambda_\pm,p,(+),t} = \frac{1}{\sqrt{2}}\big(\Psi^c_{\lambda_+,p}e^{-i\omega_+ t} + \Psi^c_{\lambda_-,p}e^{-i\omega_- t}\big) \tag{97}$$

$$= \frac{e^{-i\omega t}}{\sqrt{2}}\big(\Psi^c_{\lambda_+,p}e^{-i\Delta\omega t} + \Psi^c_{\lambda_-,p}e^{i\Delta\omega t}\big), \tag{98}$$

   where

$$\omega_+ = E_{+,B}/\hbar, \quad \omega_- = E_{-,B}/\hbar, \quad \omega = E/\hbar, \quad \Delta\omega = \Delta E/\hbar. \tag{99}$$

   This is a solution of the Schrödinger's equation with $\mathcal{H}^c_{\pi,\pm}$ as the Hamiltonian, i.e.,

$$i\hbar\frac{\partial\Psi^c_{\lambda_\pm,p,(+),t}}{\partial t} = \widehat{H}_{\pm,B}\Psi^c_{\lambda_\pm,p,(+),t}. \tag{100}$$

   We can rewrite the $\Psi^c_{\lambda_\pm,p,(+),t}$ as

$$\Psi^c_{\lambda_\pm,p,(+),t} = \frac{e^{-i\omega t}}{\sqrt{2}}\big(\Psi^c_{\lambda_\pm,p,(+)}\cos\Delta\omega t - i\Psi^c_{\lambda_\pm,p,(-)}\sin\Delta\omega t\big). \tag{101}$$

   It follows that the time evolution of $\Psi^c_{\lambda_\pm,p,t}$ is characterized by an oscillation between $\Psi^c_{\lambda_\pm,p,(+)}$ and $\Psi^c_{\lambda_\pm,p,(-)}$ [7] (p. 604).

4. It would be interesting to see if it is possible to excite the ground state $\Psi^c_{\lambda_-,p}$ into the state $\Psi^c_{\lambda_+,p}$ by injecting an amount of energy $2\Delta E$. In other words, the absorption of such an amount of energy can provide an experimental evidence for the transition from $\Psi^c_{\lambda_-,p}$ into the state $\Psi^c_{\lambda_+,p}$. The reverse processes would involve the emission of energy $2\Delta E$. An absorption of energy ΔE from the ground state $\Psi^c_{\lambda_-,p}$ would mean the transition from $\Psi^c_{\lambda_-,p}$ to the Schrödinger's cat state $\Psi^c_{\lambda_\pm,p,(+)}$ or $\Psi^c_{\lambda_\pm,p,(-)}$. A similar remark applies to emission of energy ΔE. These would provide experimental evidence for Schrödinger's cat states.

6 Concluding Remarks

Quantisation by parts in an appropriate path space allows the construction of a consistent model of the macroscopic superconducting quantum circuit of the half-line/plane geometry. The theory predicts that:

1. The circulating current on the plane can only generate a magnetic moment of magnitude half of a Bohr magneton and a magnetic flux half that of the magnetic flux quantum of a superconducting ring.
2. Under a weak magnetic field the energy difference between the two states $\Psi^c_{\lambda_-,p}$ and $\Psi^c_{\lambda_+,p}$ corresponding respectively to the supercurrent circulating in the clockwise and anti-clockwise directions in the plane is $2\Delta E = B\mu_B$.
3. There is also the possibility of a superposition of the two states of opposing circulating currents resembling a Schrödinger's cat situation. Such a superposed state, e.g., $\Psi^c_{\lambda_\pm,p,(+)}$, has an energy lying between that of $\Psi^c_{\lambda_-,p}$ and $\Psi^c_{\lambda_+,p}$. Indeed, a desire for a zero angular momentum state for the system would favour such a superposition.
4. An experimental observation on the system's energy absorption or emission can test the validity of our theory. An absorption or emission of energy of $2\Delta E$ would lend support to the existence of the states $\Psi^c_{\lambda_-,p}$ and $\Psi^c_{\lambda_+,p}$, and an absorption or emission of energy of ΔE would support the existence of their superposition. The circuit can therefore serve as a platform for experiments probing of the existence or otherwise of Schrödinger's cat states.

The need for quantisation in the path space is a direct consequence of the topology of our circuit geometry. An experimental confirmation of our results would justify both the quantisation by parts scheme and the path space formulation of quantum mechanics. The present method can be extended to more complex superconducting circuits and as well as Bose Einstein condensate.

Appendix 1: Momentum Operators in $\mathcal{H}^c$

The operator $\widehat{p}_0(\mathbb{E}^-)$ in $\mathcal{H}(\mathbb{E}^-)$ has deficiency indices (0,1) [7] (p. 127, p. 210). Its negative deficiency subspace is one-dimensional spanned by normalized function

$$\varphi_1(x) = \sqrt{\frac{2}{\hbar}}\, e^{x/\hbar}. \tag{102}$$

On the other hand $\widehat{p}_{r0}(\mathbb{E}^2_h)$ in $\mathcal{L}_n(\mathbb{E}^2_h)$ has deficiency indices (1,0). Its positive deficiency subspace is one-dimensional spanned by

$$\varphi_{2n}(r,\theta) = \sqrt{\frac{1}{\hbar\pi r}}\, e^{-r/\hbar} \otimes \eta_n(\theta). \tag{103}$$

The operator $\widehat{P}^c_{0,n}$ in $\mathcal{H}^c_n$ has deficiency indices (1,1) with its positive and negative deficiency subspaces spanned by normalized functions

$$\varphi^c_{1n}(x,r,\theta) = \sqrt{\frac{2}{\hbar}} e^{x/\hbar} \oplus 0_n(\mathbb{E}^2_h) \tag{104}$$

and

$$\varphi^c_{2n}(x,r,\theta) = 0(\mathbb{E}^-) \oplus \sqrt{\frac{1}{\hbar\pi r}} e^{-r/\hbar} \otimes \eta_n(\theta), \tag{105}$$

where $0_n(\mathbb{E}^2_h)$ is the restriction of $0(\mathbb{E}^2_h)$ to $\mathcal{L}_n(\mathbb{E}^2_h)$. These normalized functions in $\mathcal{H}^c_n$ are unitarily related [7] (p. 110). It follows from a von Neumann formula that a selfadjoint extension of $\widehat{P}^c_{\lambda_n}$ of $\widehat{P}^c_{0,n}$ must act on a domain with elements of the form [7] (pp. 128–130, p. 212)

$$\Psi^c_{\lambda_n} = \Psi^c_{0,n} + \alpha\left(\varphi^c_{2n} + e^{-i\lambda_n}\varphi^c_{1n}\right), \quad \lambda_n \in (-\pi, \pi], \tag{106}$$

where $\Psi^c_{0,n} \in \mathcal{D}(\widehat{P}^c_0) \cap \mathcal{H}^c_n$. For $\Psi^c_{\lambda_n}$ of the form

$$\Psi^c_{\lambda_n} = \phi_1(x) \oplus \phi_2(r) \otimes \eta_n(\theta), \tag{107}$$

equation (106) imposes a boundary condition at the junction on ϕ_1 and ϕ_2, i.e.,

$$\lim_{x\to 0_-} e^{i\lambda_n}\phi_1(x) = \lim_{r\to 0_+} \sqrt{2\pi r}\,\phi_2(r). \tag{108}$$

The operator $\widehat{P}^c_{\lambda_n}$ admits the following generalized eigenfunctions

$$\Psi^c_{\lambda_n,p} = e^{\dot{\imath}px} \oplus e^{i\lambda_n}\left(\frac{1}{\sqrt{2\pi r}} e^{\dot{\imath}pr} \otimes \eta_n(\theta)\right) \tag{109}$$

which satisfy the above boundary condition [7] (p. 212).

The operator $\widehat{p}_{r0}(\mathbb{E}^2_h)$ in $\mathcal{H}(\mathbb{E}^2_h)$ has a unique maximal symmetric extension $\widehat{p}_r$ in $\mathcal{H}(\mathbb{E}^2_h)$, which is often identified as the radial momentum operator in $\mathcal{H}(\mathbb{E}^2_h)$. It is of interest to note that $\widehat{p}_r$ does not admit any generalized eigenfunction on account of the boundary condition at $r = 0$ [11]; the same statement applies to $\widehat{p}_0(\mathbb{E}^2_h)$ in $\mathcal{H}(\mathbb{E}^-)$ [7] (pp. 126–127). On the other hand the composite momentum $\widehat{P}^c_{\lambda_n}$ in $\mathcal{H}^c_n$ does admit generalized eigenfunctions $\Psi^c_{\lambda_n,p}$ which satisfy boundary condition in (108). The intrinsic reason is that $\widehat{p}_r$ is only strictly maximal symmetry while $\widehat{P}^c_{\lambda_n}$ is selfadjoint. Such results can have important physical consequences on certain quantum circuits [7] (pp. 542–548). For a discussion of generalized eigenfunctions see [7] (p. 126, pp. 447–448).

Appendix 2: Hamiltonian Operators in $\mathcal{H}^c$

Restricted to subspace $\mathcal{L}_n(\mathbb{E}^2_h)$ the operator $\widehat{H}_{20}$ reduces to

$$\widehat{H}_{20,n} = -\frac{\hbar^2}{2m_c}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)_0 \otimes \widehat{\mathbb{I}}_n(\mathcal{C}). \tag{110}$$

Here $\widehat{\mathbb{I}}_n(\mathcal{C})$ is the restriction of $\widehat{\mathbb{I}}(\mathcal{C})$ to $\mathcal{L}_n(\mathcal{C})$. Define a set of new operators $\widehat{h}_{20,n}$ in $\mathcal{H}(\mathbb{R}^+)$ by

$$\widehat{h}_{20,n} = -\frac{\hbar^2}{2m_c}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)_0 \tag{111}$$

acting on $C_0^\infty(\mathbb{R}^+)$. This enables us to rewrite $\widehat{H}_{20,n}$ and $\widehat{H}_{20}$ as a direct sum within $\mathcal{H}(\mathbb{E}_h^2)$, i.e.,

$$\widehat{H}_{20,n} = \widehat{h}_{20,n} \otimes \widehat{\mathbb{I}}_n(\mathcal{C}) \quad \text{and} \quad \widehat{H}_{20} = \oplus_n \left(\widehat{h}_{20,n} \otimes \widehat{\mathbb{I}}_n(\mathcal{C})\right). \tag{112}$$

Now introduce two operators, one in $\oplus_{n\neq 0}\mathcal{L}_n(\mathbb{E}_h^2)$ and one in $L^2(\mathbb{E}^-) \oplus \mathcal{L}_0(\mathbb{E}_h^2)$:

1. In $\oplus_{n\neq 0}\mathcal{L}_n(\mathbb{E}_h^2)$: $\widehat{h} = \oplus_{n\neq 0}(\widehat{h}_{20,n} \otimes \widehat{\mathbb{I}}_n(\mathcal{C}))$, with $\bar{h}$ denoting its closure.
2. In $L^2(\mathbb{E}^-) \oplus \mathcal{L}_0(\mathbb{E}_h^2)$: $\widehat{K}_0 = \widehat{H}_{10} \oplus (\widehat{h}_{20,0} \otimes \widehat{\mathbb{I}}_0(\mathcal{C}))$, with $\widehat{K}$ denoting a selfadjoint extension.

Exner and Seba [8, 9] show that any selfadjoint extension $\widehat{H}^c$ of $\widehat{H}_0^c$ in $\mathcal{H}^c$ is of the form

$$\widehat{H}^c = \widehat{K} \oplus \bar{h}. \tag{113}$$

Moreover these selfadjoint extensions can be characterized by boundary conditions. Let $\Psi^c = \psi_1(x) \oplus \psi_2(r) \otimes \eta(\theta)$ be a member of $\mathcal{H}^c$. Define the following so-called boundary values of Ψ^c at the boundary defined by $x = 0$ and $r = 0$:

$$V_{11}(\Psi^c) = \lim_{x\to 0_-} \psi_1(x), \qquad V_{12}(\Psi^c) = \lim_{r\to 0_+} \frac{\psi_2(r)}{\ln r}; \tag{114}$$

$$V_{21}(\Psi^c) = \lim_{x\to 0_-} \frac{d\psi_1}{dx}, \qquad V_{22}(\Psi^c) = \lim_{r\to 0_+}\Big(\psi_2(r) - V_{12}(\Psi^c)\ln r\Big). \tag{115}$$

Let us call $V_{11}(\Psi^c)$ and $V_{12}(\Psi^c)$ the first set of boundary values and $V_{21}(\Psi^c)$ and $V_{22}(\Psi^c)$ the second set of boundary values. Then a selfadjoint extension $\widehat{H}^c$ of $\widehat{H}_0^c$ can be specified by its domain consisting of elements of $\mathcal{H}^c$ necessarily satisfying one of the following five classes of boundary conditions with suitably chosen numerical constants A, B, C, D [8]:

1. Class 1

$$V_{21}(\Psi^c) = AV_{11}(\Psi^c) + BV_{12}(\Psi^c), \tag{116}$$

$$V_{22}(\Psi^c) = CV_{11}(\Psi^c) + DV_{12}(\Psi^c). \tag{117}$$

In other words the second set of boundary values are expressible in terms of a linear combination of the first set of boundary values.

2. Class 2

$$V_{12}(\Psi^c) = AV_{11}(\Psi^c), \tag{118}$$

$$V_{22}(\Psi^c) = CV_{11}(\Psi^c) + DV_{21}(\Psi^c). \tag{119}$$

3. Class 3

$$V_{11}(\Psi^c) = 0, \tag{120}$$

$$V_{22}(\Psi^c) = DV_{12}(\Psi^c). \tag{121}$$

4. Class 4

$$V_{12}(\Psi^c) = 0, \tag{122}$$

$$V_{21}(\Psi^c) = A V_{11}(\Psi^c). \tag{123}$$

5. Class 5

$$V_{11}(\Psi^c) = V_{12}(\Psi^c) = 0. \tag{124}$$

For $\Psi^c_{\lambda_n,p}$ we have the following boundary values:

$$V_{11}(\Psi^c_{\lambda_n,p}) = 1, \quad V_{12}(\Psi^c_{\lambda_n,p}) = -\infty, \tag{125}$$

$$V_{21}(\Psi^c_{\lambda_n,p}) = \imath p, \quad V_{22}(\Psi^c_{\lambda_n,p}) = \infty, \tag{126}$$

where the infinity sign for $V_{12}(\Psi^c_{\lambda_n,p})$ and $V_{12}(\Psi^c_{\lambda_n,p})$ symbolizes a complex number of infinite magnitude. Clearly these boundary values violate each of the above classes of boundary conditions. This shows that $\Psi^c_{\lambda_n,p}$ cannot be a generalized eigenfunction of the any selfadjoint extension of $\mathcal{H}^c_0$.

Appendix 3: Momentum Operators in $\mathcal{H}^c_{\gamma,n}$

Operator $\widehat{P}^c_{0\gamma,n}$ in $\mathcal{H}^c_{\gamma,n}$ has deficiency indices (1,1) with its positive and negative deficiency subspaces spanned by normalized functions

$$\phi^c_{2n}(x,r,\theta) = 0(\mathbb{E}^-) \oplus \sqrt{\frac{1}{\hbar\pi r}} e^{-r/\hbar} \otimes \eta_{\gamma,n}(\theta_{ex}), \tag{127}$$

and

$$\phi^c_{1n}(x,r,\theta) = \sqrt{\frac{2}{\hbar}} e^{x/\hbar} \oplus 0_n(\Pi(\mathbb{E}^2_h)), \tag{128}$$

where $0_n(\Pi(\mathbb{E}^2_h))$ is the restriction of the zero element $0(\Pi(\mathbb{E}^2_h)) \in \mathcal{H}_\gamma(\Pi(\mathbb{E}^2_h))$ to $\mathcal{H}_{\gamma,n}(\Pi(\mathbb{E}^2_h))$. The arguments in Appendix 1 apply to obtain the selfadjoint extensions of $\widehat{P}^c_{0\gamma,n}$ in $\mathcal{H}^c_{\gamma,n}$.

Appendix 4: Generalized Eigenfunctions and Normalization

The expectation value of an observable $\widehat{A}$ with respect to an unnormalized vector $\psi(x)$ in the domain of $\widehat{A}$ is given by

$$\langle \psi \mid \widehat{A}\psi \rangle = \lim_{x_a \to -\infty, x_b \to \infty} \frac{\int_{x_a}^{x_b} \psi^*(x) \widehat{A}\psi(x) dx}{\int_{x_a}^{x_b} |\psi(x)|^2 dx}. \tag{129}$$

This expression can be extended to generalized eigenfunctions of an observable which are not normalizable in order to retain a formal concept of expectation values for generalized eigenfunctions. To illustrate such a *normalization procedure* take the simple example of the momentum operator $-i\hbar d/dx$ in the Hilbert space $L^2(\mathbb{E})$ with the plane waves $\phi(x) = \exp(ipx)$ as generalized eigenfunctions. We can define the expectation value as

$$\langle \phi \mid (-i\hbar d/dx)\phi \rangle = \lim_{x_a\to-\infty, x_b\to\infty} \frac{\int_{x_a}^{x_b} \phi^*(x)(-i\hbar d/dx)\phi(x)dx}{\int_{x_a}^{x_b} |\phi(x)|^2 dx} = p. \tag{130}$$

For a global quantity in our two-branch circuit we have to carry out the normalization procedure on the two branches separately. Each branch will formally produce a value. The expectation value for the circuit as a whole would be the sum of the values in the two branches divided by two. So, for the linear momentum $\widehat{P}^c_{\gamma,\lambda_n}$ in $\Psi^c_{\gamma,\lambda_n,p}$ the expectation value is given by

$$\langle \Psi^c_{\gamma,\lambda_n,p} \mid \widehat{P}^c_{\gamma,\lambda_n} \Psi^c_{\gamma,\lambda_n,p} \rangle \tag{131}$$

$$= \frac{1}{2}\left(\lim_{x_a\to-\infty, x_b\to 0} \frac{\int_{x_a}^{x_b} \phi^*(x)(-i\hbar d/dx)\phi(x)dx}{\int_{z_a}^{z_b} |\phi(x)|^2\, dx} \right. \tag{132}$$

$$\left. + \lim_{r_a\to 0, r_b\to\infty} \frac{\int_{r_a}^{r_b}\int_0^{2\pi} \psi^*_{\gamma,n,p}(r,\theta_{ex})[-i\hbar(\partial/\partial r + 1/2r)]\psi_{\gamma,n,p}(r,\theta_{ex}) r dr d\theta_{ex}}{\int_{r_a}^{r_b}\int_0^{2\pi} |\psi_{\gamma,n,p}(r,\theta_{ex})|^2 r dr d\theta_{ex}} \right) \tag{133}$$

$$= p. \tag{134}$$

For a quantity localized in the plane the normalization procedure is also needed, albeit only on the plane. An example is the magnetic moment generated by a circulating current on the plane. This is a quantity localized in the plane. Integrating the element of magnetic moment in (74) produces an infinite result. A finite result is obtained with the normalized expression in (75). A similar argument applies to the calculation of the magnetic flux.

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